ALGEBRA.

103. Proposed by WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.

Given the equation $x^m+p_1x^{m-2}+p_2x^{m-2}+\ldots p_{m-1}x+p_m=0$ freed from multiple roots. Prove that its discriminant is positive or negative according as the number of pairs of complex roots is even or odd.

I. Solution by the PROPOSER.

We have
$$\begin{aligned}\triangle=&(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2\ldots(x_1-x_m)^2\\ &(x_2-x_3)^2(x_2-x_4)^2\ldots(x_2-x_m)^2\\ &(x_3-x_4)^2\ldots(x_3-x_m)^2\\ &\ldots\ldots\ldots\ldots\ldots\\ &(x_{m-1}-x_m)^2.\end{aligned}$$

There are seven cases to be considered :

I. Differences of real roots.
II. Differences of a real and a complex root.
III. Differences of a real and a pure imaginary root.
IV. Differences of two pure imaginaries.
V. Differences of a pure imaginary and a complex.
VI. Differences of two complex roots not conjugate.
VII. Differences of two conjugate complex roots.

I. It is evident at once, since each difference is squared, that the real roots alone can not affect the sign.

II. If c be a real root, and $a+bi$ a complex, for every difference of the form $c-(a+bi)$ we shall also have one of the form $c-(a-bi)$ and as the product of these two is positive, their square is also positive, and hence case II can not affect the sign of $\triangle$.

By exactly the same reasoning it can be shown that cases III, V, and VI cannot affect the sign of $\triangle$.

IV is a special form of VII, so we shall pass to the last. Let $a+bi$ and $a-bi$ be two conjugate complex roots. Then the difference is $\pm bi$ whose square is $-b$. Hence for every pair of roots we use in forming a difference, we change the sign of $\triangle$, and hence $\triangle$ is positive or negative according as the number of pairs of complex roots is even or odd.

II. Solution by J. W. YOUNG, Fellow and Assistant in Mathemetics, Ohio State University, Columbus, O.

The discriminant is equal to the product of squares of the differences of roots taken in all possible combinations, *i. e.*: $\triangle=\Pi(x_1-x_2)^2$ $[x, x_2, x_3\ldots x_n$ $=$roots of equation].

For every complex pair of roots, there will be a factor of the form

$$[(\xi+i\eta)-(\xi-i\eta)]^2=[2i\eta]^2=-4\eta^2.$$

That is, for every complex pair there will be a *negative* factor in the discriminant.

Moreover the remaining factors will give you a *positive* product.

For, if one of the factors be

$$[(\xi_1+i\eta_1)-(\xi_2+i\eta)]^2=[(\xi_1-\xi_2)+i(\eta_1-\eta_2)]^2$$

there will be a corresponding factor

$$[(\xi_1-i\eta_1)-(\xi_2-i\eta_2)]^2=[(\xi_1-\xi_2)-i(\eta_1-\eta_2)]^2.$$

When multiplied these give

$$[(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2]^2,$$

which is clearly positive. So all the factors containing complex roots may be taken in pairs which give a positive product, and, of course, the factors containing real roots only are positive. Hence corresponding to every pair of complex roots there will be *one* and *only one* negative factor. Therefore, if the number of pairs be even the discriminant will be positive; if odd, negative.

NOTE.=Mr. Harry S. Vandiver should have been credited as a joint author of the solution of problem 102.

GEOMETRY.

129. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that at no point of an ellipse will the circle of curvature pass through the center, if the eccentricity be less than $\frac{1}{2}\sqrt{2}$.

I. Solution by F. ANDEREGG, A.M., Professor of Mathematics, Oberlin College, Oberlin, O.; W.H. CARTER, A. M., Professor of Mathematics, Centenary College, Jackson, La.

Since for the ellipse the radius of curvature is

$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4};$$

and the center of curvature is the point

$$\left(\frac{(a^2-b^2)x^3}{a^4},\ -\frac{(a^2-b^2)y^3}{b^4}\right),$$

the equation of the circle of curvature is

$$\left(x'-\frac{(a^2-b^2)x^3}{a^4}\right)^2+\left(y'+\frac{(a^2-b^2)y^3}{b^4}\right)^2=\frac{(a^4y^2+b^4x^2)^3}{a^8b^8}.$$

This circle passes through the origin

$$(a^2-b^2)^2(b^8x^6+a^8y^6)=(a^4y^2+b^4x^2)^3.$$